Calculus IV
Exam I

Fall 09 (November 5, 09)

Name: Solution

1. Find the points on the sphere $x^2 + y^2 + z^2 = 1$ where the tangent plane is parallel to the plane $2x + y - 3z = 2$.

$\vec{N} \parallel \vec{\nabla f}$

12%

$\vec{\nabla f} = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}$ Must be $= c\vec{N}$

$= 2c\,\vec{i} + c\,\vec{j} - 3c\,\vec{k}$

$\Rightarrow$ Must have $\begin{cases} 2x = 2c \\ 2y = c \\ 2z = -3c \\ x^2 + y^2 + z^2 = 1 \end{cases}$ $c = x =$ $2y = \frac{-2}{3} z$

$c^2 + \frac{c^2}{4} + \frac{9c^2}{4} = 1$

$\Rightarrow c = \pm 1$

$(\pm 1, \pm \frac{1}{2}, \mp \frac{3}{2})$.

2. Consider the function $z = f(x, y) = x^2 - xy + 3y^2$.

12%

(a) Estimate $f(2.96, -0.95)$

$(3, -1)$: closest point.

$f(3, -1) = 15$

$\Delta x = -0.04$

$\Delta y = -0.95 + 1 = 0.05$

$f_x = 2x - y = 7$

$f_y = -x + 6y = -$

$L(x, y) = 15 + 7\Delta x - 9\Delta y$

$\Rightarrow f(2.96, -0.95) \approx \boxed{15 + 7(-0.04) - 9(0.05)}$

(b) If (x, y) changes from $(3, -1)$ to $(2.96, -0.95)$, compare the values of Δz (the exact change) and dz. No need to complete your computations

5%

$\Delta z =$ Exact difference

$= f(2.96, -0.95) - f(3, -1)$

$\Delta\, |dz| = |f(3, -1) - L(2.96, -0.95)|$

(c) Write the equation of the tangent plane at $(3, -1)$.

6%

$z = z_0 + f_x\, \Delta x + f_y\, \Delta y$

$\boxed{z = 15 + 7(x - 3) - 9(y + 1)}$

3. Use differentials to approximate the amount of tin needed to construct a box of sides $10cm$ each and of thickness $0.5cm$ per face.

12%

$V = x^3$

$\Delta V = 3x^2 dx.$

$= 3(10)^2 \cdot (0.5)$

4. Without finding the equation of the level curve, how would you find the slope of the tangent line to the curve obtained by intersecting the surface $z = f(x,y)$ with the plane $x = 5$. Same for intersecting the surface $z = f(x,y)$ with the plane $y = 3$.

8%.

$f_y(5,3)$

$f_y(5,3)$

5. Find the maximum an minimum of the function: $z = f(x,y) = x^2 + xy + y^2 - 6x + 2$ over the rectangle: $0 \leq x \leq 5, -3 \leq y \leq 0$

16%.

$f_x = 2x + y - 6 = 0$ $\quad \begin{cases} 2x + y = 6 \\ x + 2y = 0 \end{cases}$ $\quad -3y = 6$

$f_y = x + 2y = 0$ $\quad y = -2$

$x = 4$

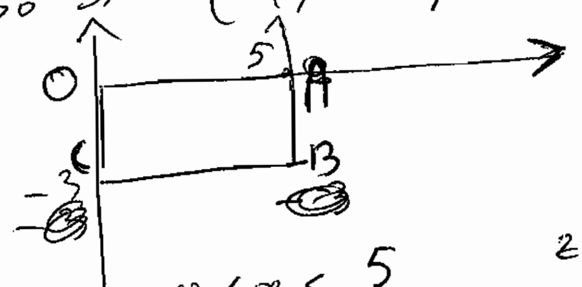
$(4,-2)$: crit. point

$f_{xy} = 1 \quad f_{xx} = 2 \quad f_{yy} = 2$

$\Delta = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 > 0.$

$f_{xx} > 0 \Rightarrow (4,-2)$: min (local)?

The Rect.



1) On OA: $y = 0 \quad 0 \leq x \leq 5$ $\quad z = x^2 - 6 + 2$

$z' = 2x - 6 \quad x = 3$

$\boxed{(3,0)}$

2) On OC: $x = 0 \quad -3 \leq y \leq 0 \quad z = y^2 + 2 \quad z' = 2y$

$\boxed{(0,0)}$

3) On AB: $x = 5$

$z = 25 + 5y + y^2 - 30 + 2$

$z = y^2 + 5y - 3$ $\quad z' = 2y + 5 \quad y = -5/2$

$(5, -5/2)$

4) On CB: $y = -3 \quad 0 < x < 5$ $\quad z = x^2 - 3x + 9 - 6x + 2$

$= x^2 - 9x + 11$

$\boxed{(9/2, -3)}$ $\quad z' = 2x - 9 \quad x = 9/2$

Candidate: $(9/2, -3), (5, -5/2); (0,0), (3,0)\ (4,-2)$

6. Find the point on the paraboloid $z = \frac{x^2}{5} + \frac{y^2}{4}$ that is closest to the point $(0,5,0)$.

Minimize $f = x^2 + (y-5)^2 + z^2$ subject to:

$$g = \frac{x^2}{5} + \frac{y^2}{4} - z = 0.$$

$\nabla f = \lambda \nabla g$

$$\begin{cases} 2x = \lambda . \frac{2x}{5} \\ 2(y-5) = \lambda . \frac{2y}{4} \\ 2z = -1 \\ \frac{x^2}{5} + \frac{y^2}{4} - z = 0 \end{cases}$$

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7. Consider the function $f(x,y,z) = 3x^2 + 2y^2 - 4z$

(a) Find the instantaneous rate of change ~~of~~ δf at the point $(-1,-3,2)$ in the direction from P to the point $Q(-4,1,-2)$.

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$$\vec{u} = \frac{(-1+4)i + (-3-1)j + (2+2)k}{\sqrt{9+16+16}} = \frac{3\vec{i} - 4\vec{j} + 4\vec{k}}{\sqrt{41}}$$

$$\vec{\nabla f} = 6x\vec{i} + 4y\vec{j} - 4\vec{k}\Big|_{(-1,-3,-2)}$$

$$= -6i - 12j - 4\vec{k}$$

$$D_{\vec{u}} f = (-6i - 12j - 4k) \cdot \left(\frac{3i - 4j + 4k}{\sqrt{41}}\right)$$

$$= \frac{-18 + 48 - 16}{\sqrt{41}} = \frac{14}{\sqrt{41}}$$

(b) In which direction will the change be maximum at P?

4%

In the dir of $\vec{\nabla f}$

$$\vec{V} = \frac{-6i - 12j - 4\vec{k}}{\sqrt{36 + 144 + 16}}$$